Tutorial 9

March 31, 2016

1. **Term by term integration of Fourier series**

   (a) If $f(x)$ is a piecewise-continuous function on $[-l, l]$, show that $F(x) = \int_{-l}^{x} f(s)ds$ has a full Fourier series that converges pointwise.

   (b) If $f(x)$ is a piecewise-continuous function on $[-l, l]$, show that

$$F(x) = \int_{-l}^{x} f(s)ds = \frac{a_0}{2}(x + l) + \frac{l}{n\pi}\sum_{n=1}^{\infty} a_n \sin\frac{n\pi x}{l} - b_n \cos\frac{n\pi x}{l} + (-1)^n b_n$$

   where $a_0, a_n, b_n, n = 1, 2, \cdots$ are Fourier coefficients of $f(x)$ which is given by

$$a_n = \frac{1}{l}\int_{-l}^{l} \cos\frac{n\pi x}{l} f(x)dx, \quad n = 0, 1, \cdots$$

$$b_n = \frac{1}{l}\int_{-l}^{l} \sin\frac{n\pi x}{l} f(x)dx, \quad n = 1, 2, \cdots$$

Solution:

(a) First $F(x)$ is continuous on $[-l, l]$. Second, $F'(x) = f(x)$ is piecewise-continuous on$[-l, l]$. Thus the full Fourier series of $F(x)$ converges pointwise to $F(x)$ on $(-l.l)$.

(b) Consider $G(x) = F(x) - \frac{a_0}{2}(x + l)$. Note that $a_0 = \frac{1}{l}\int_{-l}^{l} f(x)dx = \frac{1}{l}F(l)$, we have $G(-l) = G(l) = 0$. Then we can expand $G(x)$ as a continuous function of period $2l$. $G'(x) = f(x) - \frac{a_0}{2}$ is piecewise-continuous on whole line. Thus the full Fourier series of $G(x)$ converges pointwise to $G(x)$ for $-\infty < x < \infty$.
The full Fourier series of $G(x)$ is

$$G(x) = \frac{1}{2}A_0 + \sum_{n=1}^{\infty}(A_n \cos\frac{n\pi x}{l} + B_n \sin\frac{n\pi x}{l})$$

where the coefficients are

$$\begin{aligned} A_n =& \frac{1}{l}\int_{-l}^{l} G(x)\cos(\frac{n\pi x}{l})dx \\ =& \frac{1}{n\pi}\sin(\frac{n\pi x}{l})G(x)\Big|_{-l}^{l} - \frac{1}{n\pi}\int_{-l}^{l}\sin(\frac{n\pi x}{l})(f(x) - \frac{a_0}{2})dx \\ =& -\frac{1}{n\pi}\int_{-l}^{l}\sin(\frac{n\pi x}{l})f(x)dx \\ =& -\frac{l}{n\pi}b_n, \quad n = 1, 2, \cdots \end{aligned}$$

$$
\begin{aligned}
B_n =& \frac{1}{l}\int_{-l}^{l} G(x)\sin(\frac{n\pi x}{l})dx \\
=& -\frac{1}{n\pi}\cos(\frac{n\pi x}{l})G(x)\Big|_{-l}^{l} + \frac{1}{n\pi}\int_{-l}^{l}\cos(\frac{n\pi x}{l})(f(x)-\frac{a_0}{2})dx \\
=& \frac{1}{n\pi}\int_{-l}^{l}\cos(\frac{n\pi x}{l})f(x)dx \\
=& \frac{l}{n\pi}a_n, \quad n = 1, 2, \cdots
\end{aligned}
$$

For the coefficient A_0, by taking $x = -l$, we have $0 = G(l) = \frac{A_0}{2} + \sum_{n=1}^{\infty} A_n(-1)^n$. Hence

$$
F(x) - \frac{a_0}{2}(x+l) = \frac{l}{n\pi}\sum_{n=1}^{\infty} a_n \sin\frac{n\pi x}{l} - b_n\cos\frac{n\pi x}{l} + (-1)^n b_n
$$

2. **Shifting data method**

Consider the following inhomogeneous wave equations:

$$
\begin{cases}
u_{tt} - c^2 u_{xx} = F(x)\cos wt \\
u(0,t) = H\cos wt, \qquad u(l,t) = K\cos wt \\
u(x,0) = \phi(x), \qquad u_t(x,0) = \psi(x)
\end{cases} \tag{1}
$$

We wish to subtract a solution of

$$
\begin{cases}
U_{tt} - c^2 U_{xx} = F(x)\cos wt \\
U(0,t) = H\cos wt, \qquad U(l,t) = K\cos wt
\end{cases}
$$

A good guess is that U should have the form $U(x,t) = u_0(x)\cos wt$, thus $u_0(x)$ satisfies

$$
\begin{cases}
-w^2 u_0 - c^2 u_0'' = F(x) \\
u_0(0) = H, \qquad u_0(l) = K
\end{cases}
$$

This is a solvable second order ODE. Thus we can find a special solution $U(x,t) = u_0(x)\cos wt$.
Let u be a solution to (1), set $v = u - U$, then v satisfies

$$
\begin{cases}
v_{tt} - c^2 v_{xx} = 0 \\
v(0,t) = 0, \qquad v(l,t) = 0 \\
v(x,0) = \phi(x) - u_0(x), \quad u_t(x,0) = \psi(x)
\end{cases} \tag{2}
$$

This is a solvable homogeneous wave problem which we can use the seperation of variables, for example.

3. **Invariance Properties of Δ_3**

   (a) Δ_3 is invariant under translation.

   (b) Δ_3 is invariant under rotation.

   Any rotation in three dimensions is given by

$$
x' = Ox
$$

where $O = (o_{ij})$ is an othogonal matrix, that is, $O^t O = OO^t = I$. Therefore,

$$
\begin{aligned}
\Delta u =& \sum_{i,j=1}^{3}\delta_{ij}u_{ij} = \sum_{i,j=1}^{3}\delta_{ij}\partial_i(\sum_{k=1}^{3}u_{x'_k}\frac{dx'_k}{dx_i}) = \sum_{i,j=1}^{3}\delta_{ij}\partial_i(\sum_{k=1}^{3}u_{x'_k}o_{ki}) = \sum_{i,j=1}^{3}\delta_{ij}\sum_{k,l=1}^{3}u_{x'_k x'_l}o_{ki}o_{lj} \\
=& \sum_{i,k,l=1}^{3}u_{x'_k x'_l}o_{ki}o_{li} = \sum_{k,l=1}^{3}u_{x'_k x'_l}\delta_{kl} = \Delta' u
\end{aligned}
$$

where we have used $\sum_{i=1}^{3} o_{ki}o_{li} = \delta_{kl}$.

(c) For the three-dimensional laplacian

$$\Delta_3 = \partial_x^2 + \partial_y^2 + \partial_z^2$$

it is natural to use spherical coordinates (r, θ, ϕ). First, consider the chain of variables $(x, y, z) \to (s, \phi, z)$ which is given by

$$x = s\cos\phi$$

$$y = s\sin\phi$$

$$z = z$$

By the two-dimensional Laplace calculation, we have

$$u_{xx} + u_{yy} = u_{ss} + \frac{1}{s}u_s + \frac{1}{s^2}u_{\phi\phi}.$$

Second, consider the chain of variables $(s, \phi, z) \to (r, \phi, \theta)$ which is given by

$$s = r\sin\theta$$

$$z = r\cos\theta$$

$$\phi = \phi$$

By the two-dimensional Laplace calculation, we have

$$u_{ss} + u_{zz} = u_{rr} + \frac{1}{r}u_r + \frac{1}{r^2}u_{\theta\theta}.$$

Thus we have

$$\Delta_3 u = u_{xx} + u_{yy} + u_{zz} = \frac{1}{s}u_s + \frac{1}{s^2}u_{\phi\phi} + u_{rr} + \frac{1}{r}u_r + \frac{1}{r^2}u_{\theta\theta}.$$

And note that $s = r\sin\theta$ and $u_s = u_r\frac{\partial r}{\partial s} + u_\theta\frac{\partial\theta}{\partial s} = u_r\frac{s}{r} + u_\theta\frac{\cos\theta}{r}$. Therefore

$$\Delta_3 u = \frac{1}{r^2}\cot\theta u_\theta + \frac{1}{r^2\sin^2\theta}u_{\phi\phi} + u_{rr} + \frac{2}{r}u_r + \frac{1}{r^2}u_{\theta\theta}.$$

(d) Now we want to find a solution which don't change under rotations, that is, which depend only on r. Thus

$$0 = \Delta_3 u = u_{rr} + \frac{2}{r}u_r.$$

So $(r^2 u_r)_r = 0$. It has the solutions $r^2 u_r = c_1$. That is, $u = -c_1\frac{1}{r} + c_2$. The important harmonic function

$$u(r) = \frac{1}{r} = (x^2 + y^2 + z^2)^{-\frac{1}{2}}$$

is called the fundamental solution of $\Delta_3 u = 0$.